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CONTRACTIONS ON L_1-SPACES

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Abstract. It is shown that a linear contraction on a complex L_1-space can be represented in terms of its linear modulus. This result is then used to give a direct proof of Chacon's general ratio ergodic theorem.

1. **Introduction.** The main purpose of this paper is to obtain a representation (Theorem 3.1) of a linear contraction on a complex L_1-space in terms of its linear modulus [6]. This result is especially suitable to give a rather transparent proof of a general ratio ergodic theorem of Chacon (Theorem 4.1) [4], [5] and to obtain some properties (Theorem 4.2 and Theorem 4.3) of ratio ergodic limits.

The properties of positive contractions and the ergodic theory of these contractions are assumed to be known. The results that are used in this paper, however, are stated with references. §2 includes many of these results and also some elementary lemmas. In the next section the representation theorem (Theorem 3.1) is proved and §4 contains the application of this theorem to the theory of ratio ergodic limits. Finally the last section is devoted to a different approach to the results of this paper. Here the main result is a direct proof of Theorem 4.4, which is a corollary of Theorem 4.3, but, at the same time, implies Theorem 4.1. In this section, however, many details are omitted, except in the proofs of Lemma 5.6 and Theorem 5.2, that seem to be of some intrinsic interest.

2. **Definitions and preliminaries.** Let $(X, \mathscr{F}, \mu)$ be a σ-finite measure space and $L_p=L_p(X)=L_p(X, \mathscr{F}, \mu)$, $1\leqq p\leqq\infty$, the usual (complex) Banach spaces. If $A\in\mathscr{F}$ then χ_A is the characteristic function of A and $L_p(A)$ denotes the Banach space of all L_p-functions that vanish a.e. on $A^c=X-A$; also, $L_p^+(A)$ is the set of all nonnegative $L_p(A)$-functions.

Let $T\colon L_p\to L_p$ be a (linear) operator. It is called a positive operator if $TL_p^+\subset L_p^+$ and a contraction if $\|T\|\leqq 1$. If $T\colon L_1\to L_1$ is a contraction, it is known [6] that there is then a unique positive contraction $\tau\colon L_1\to L_1$ called the linear modulus of T, so that $|Tf|\leqq\tau|f|$ for all $f\in L_1$ and that $\tau p=\sup_{|g|\leqq p}|Tg|$ for all $p\in L_1^+$.

For the remainder of this paper $T\colon L_1\to L_1$ will denote a contraction on L_1, and $\tau\colon L_1\to L_1$ will be its linear modulus. Also, T^* and τ^* will denote the corresponding adjoint operators on $L_1^*=L_\infty$. Note that T^* and τ^* are contractions on L_∞ and that τ^* is positive.

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LEMMA 2.1. *If $h \in L_\infty$ then $|T^*h| \leqq \tau^*|h|$.*

Proof. Given $h \in L_\infty$, find $s \in L_\infty$ so that $|s|=1$ and that $sT^*h = |T^*h|$. Then, for any $p \in L_1^+$,

$$\begin{aligned} \int p|T^*h|\,d\mu &= \int psT^*h\,d\mu = \int T(ps)\cdot h\,d\mu \\ &\leqq \int |Tps|\,|h|\,d\mu \leqq \int \tau|ps|\,|h|\,d\mu \\ &= \int \tau p\cdot|h|\,d\mu = \int p\tau^*|h|\,d\mu, \end{aligned}$$

meaning that $|T^*h| \leqq \tau^*|h|$.

Recall that a positive contraction $\tau: L_1 \to L_1$ partitions the measure space X into two disjoint sets C and D, called respectively, the conservative and dissipative parts of X, such that if $p \in L_1$ and $p>0$ a.e. then $\sum_{n=0}^\infty \tau^n p$ is infinite a.e. on C and is finite a.e. on D [8], [3]. The transformation τ is called conservative if $\mu(D)=0$. A set $A \in \mathscr{F}$ is said to be (τ-) invariant if $\tau L_1(A) \subset L_1(A)$, or equivalently, if $\tau^*L_\infty(A^c) = L_\infty(A^c)$. It is known that C is invariant and the class $\mathscr{I}$ of all invariant subsets of C forms a σ-field in the class of all (measurable) subsets of C [3]. Finally, if $h \in L_\infty$ then $\tau^*h = h$ a.e. on C if and only if h has an $\mathscr{I}$-measurable restriction to C [3]. In particular, $\tau^*\chi_C = \chi_C$ a.e. on C, or equivalently, $\int \tau f\,d\mu = \int f\,d\mu$ for all $f \in L_1(C)$.

LEMMA 2.2. *If $h \in L_\infty$ is real valued and if $h \leqq \tau^*h$ then $h = \tau^*h$ a.e. on C.*

Proof. Let $p \in L_1$ and $p>0$ a.e. If the conclusion of the lemma is not correct then $\int (\sum_{n=0}^\infty \tau^n p)(\tau^*h - h)\,d\mu = \infty$. But this means that $\lim_{n\to\infty} \int p(\tau^{*n}h - h)\,d\mu = \infty$, which is a contradiction.

LEMMA 2.3. *Let $A \in \mathscr{I}$ and $h \in L_\infty$. Then $\tau^*h = h$ a.e. on A if and only if $\chi_A h$ has an $\mathscr{I}$-measurable restriction to C.*

Proof. Since A is invariant, $\chi_A\tau^*h = \chi_A\tau^*(\chi_A h)$; and since $C-A$ is also invariant, $\chi_A\tau^*(\chi_A h) = \chi_C\tau^*(\chi_A h)$. Hence $\chi_C\tau^*(\chi_A h) = \chi_A\tau^*(\chi_A h) = \chi_A\tau^*h = \chi_A h = \chi_C(\chi_A h)$, i.e. $\tau^*(\chi_A h) = \chi_A h$ a.e. on C. Then the proof follows from the remarks preceding Lemma 2.2.

LEMMA 2.4. *Let $A \in \mathscr{I}$ and $h \in L_\infty$. Then $\tau(hf) = h\cdot\tau f$ for all $f \in L_1(A)$ if and only if $\tau^*h = h$ a.e. on A.*

Proof. Let $h \in L_\infty$ and $\tau(hf) = h\tau f$ for all $f \in L_1(A)$. Then $f \in L_1(A)$ $(\subset L_1(C))$ implies that $\int \tau(hf)\,d\mu = \int h\tau f\,d\mu = \int f\tau^*h\,d\mu$, i.e. $h = \tau^*h$ a.e. on A, since $\int \tau(hf)\,d\mu = \int hf\,d\mu$.

Conversely let $\mathscr{M}$ be the class of all L_∞-functions h such that $\tau(hf) = h\tau f$ for all $f \in L_1(A)$. First note that if $B \in \mathscr{I}$, $B \subset A$ and $f \in L_1(A)$ then $\tau(\chi_B f) = \chi_B\tau(\chi_B f) = \chi_B\tau(f - \chi_{A-B}f) = \chi_B\tau f$, which means that $\chi_B \in \mathscr{M}$. Since $\mathscr{M}$ is a closed linear manifold of L_∞ it then follows that $\mathscr{M}$ contains any L_∞-function h such that $\chi_A h$ has a $\mathscr{I}$-measurable restriction to C. Hence, by Lemma 2.3, $\mathscr{M}$ contains any function $h \in L_\infty$ satisfying $\tau^* h = h$ a.e. on A.

LEMMA 2.5. *If $h \in L_\infty$ and $T^*h = h$ then $\tau^*|h| = |h|$ a.e. on C.*

Proof. By Lemma 2.1, $|h| = |T^*h| \leqq \tau^*|h|$, and hence $|h| = \tau^*|h|$ a.e. on C, by Lemma 2.2.

Finally we note an elementary result on integration.

LEMMA 2.6. *Let f and g be two (measurable) functions on X such that $\int (g-f)\, d\mu = 0$ and that $|f| \leqq g$ a.e. Then $f = g$ a.e.*

Proof. Since $\operatorname{Re}(g-f) = g - \operatorname{Re} f \geqq 0$ a.e., $\operatorname{Re} \int (g-f)\, d\mu = 0$ implies that $g - \operatorname{Re} f = 0$ a.e. and hence $g = f$ a.e.

3. **The main result.** The representation of T to be given in Theorem 3.1 depends on the following two lemmas.

LEMMA 3.1. *Assume that τ is conservative and that there exists a function $h \in L_\infty$, such that $h \neq 0$ a.e. and $T^*h = h$. Let $s = h/|h|$. Then $\tau f = sT(\bar{s}f)$ for any $f \in L_1$, where $\bar{s} = 1/s$ is the complex conjugate of s.*

Proof. First note that $\tau^*|h| = |h|$ by Lemma 2.5. Also, it is enough to prove this lemma for a nonnegative L_1-function p. Now, if $p \in L_1^+$, then

$$\int \tau p \cdot |h|\, d\mu = \int p|h|\, d\mu = \int \bar{s}ph\, d\mu = \int T(\bar{s}p)h\, d\mu = \int sT(\bar{s}p)|h|\, d\mu,$$

and hence $\tau p = sT(\bar{s}p)$ by Lemma 2.6, since $|sT(\bar{s}p)| = |T(\bar{s}p)| \leqq \tau|\bar{s}p| = \tau p$.

LEMMA 3.2. *Let $A \in \mathscr{I}$ be an invariant subset of C and assume that there exists a function $h \in L_\infty$ such that $h \neq 0$ a.e. on A and such that $T^*h = h$ a.e. on A. Let $s \in L_\infty$ be any function such that $s = h/|h|$ a.e. on A. Then $\tau f = sT(\bar{s}f)$ for all $f \in L_1(A)$.*

Proof. Since A is an invariant set, $\tau L_1(A) \subset L_1(A)$ and hence $TL_1(A) \subset L_1(A)$. Let τ_A and T_A be, respectively, the restrictions of τ and T to the L_1-space $L_1(A)$. Also, for any function g on X, let $g_A = \chi_A g$ be the restriction of this function to A. Then, clearly, τ_A is the linear modulus of T_A, τ_A is conservative and $T^*h_A = h_A$ with $h_A \neq 0$ a.e. on A. Hence the previous lemma applies and gives that $\tau_A f = s_A T_A(\bar{s}_A f)$ for all $f \in L_1(A)$. Since $\tau_A f = \tau f$ and $T_A(\bar{s}_A f) = T(\bar{s}f)$ for such a function f, this completes the proof of the present lemma.

THEOREM 3.1. *Let T be a contraction on L and let τ be its linear modulus. Let C denote the conservative part of X with respect to τ and let $\mathscr{I}$ be the σ-field of invariant subsets of C. Then there exists a set $\Gamma \in \mathscr{I}$ and a function $s \in L_\infty(\Gamma)$ such that*

(i) *$|s| = 1$ a.e. on Γ and $Tf = \bar{s}\tau(sf)$ for any $f \in L_1(\Gamma)$,*

(ii) *if $\Delta = C - \Gamma$ then $(1-T)L_1(\Delta)$ is dense in $L_1(\Delta)$, in the norm topology,*

(iii) *the partition of C into Γ and Δ is unique (up to sets of measure zero),*

(iv) *a function r satisfies the properties stated above for s if and only if there exists a function $l = L_\infty(\Gamma)$ such that $|l| = 1$ a.e. on Γ, $\tau^* l = l$ a.e. on Γ and $r = sl$.*

Proof. Let $\mathscr{A}$ be the class of all sets $A \in \mathscr{I}$ for which there exists a function $h \in L_\infty$ such that $h \neq 0$ a.e. on A and $T^*h = h$ a.e. on A. Let ν be any finite measure on $(X, \mathscr{F})$, equivalent to μ, and $\alpha = \sup \{\nu(A) \mid A \in \mathscr{A}\}$. Let $A'_n \in \mathscr{A}$ and $\lim_{n\to\infty} \nu(A'_n) = \alpha$. Then $A_1 = A'_1$, $A_{n+1} = A'_{n+1} - (\bigcup_{k=1}^n A_k)$, $n \geqq 1$, defines a sequence of pairwise disjoint sets in $\mathscr{A}$ such that $\nu(\bigcup_{n=1}^\infty A_n) = \alpha$. Let $\Gamma = \bigcup_{n=1}^\infty A_n$ and let $s \in L_\infty(\Gamma)$ be defined as $s = h_n/|h_n|$ a.e. on A_n, where $h_n \in L_\infty$, $h_n \neq 0$ a.e. on A_n and $T^*h_n = h_n$ a.e. on A_n.

Then the proof of (i) follows immediately from Lemma 3.2.

For the proof of (ii), consider a bounded linear functional on $L_1(\Delta)$ that vanishes on $(1-T)L_1(\Delta)$. If this functional is represented by (the restriction to Δ of) a function $h \in L_\infty$ then $\int (f - Tf)h \, d\mu = 0$ for all $f \in L_1(\Delta)$, which means that $h = T^*h$ a.e. on Δ. Hence, by the definition of Γ, $h = 0$ a.e. on Δ. Therefore any bounded linear functional vanishing on $(1-T)L_1(\Delta)$ vanishes also on $L_1(\Delta)$. This proves (ii).

For the uniqueness of Γ and Δ, first note that if $B \in \mathscr{I}$ and $B \subset \Delta$ then $(1-T)L_1(B)$ is dense in $L_1(B)$, in the norm topology. Since the argument given in the preceding paragraph is reversible, this means that if $h \in L_\infty$ and $h = T^*h$ a.e. on B then $h = 0$ a.e. on B. It is then easy to see that this implies (iii).

Finally let $r \in L_\infty(\Gamma)$ and $|r| = 1$ a.e. on Γ. Define $l \in L_\infty(\Gamma)$ so that $r = sl$. Then $|l| = 1$ a.e. on Γ. To prove (iv) it is enough to show that $Tf = \bar{r}\tau(rf)$ for all $f \in L_1(\Gamma)$ if and only if $\tau^*l = l$ a.e. on Γ. But $\bar{r}\tau(rf) = \bar{s}\bar{l}\tau(slf)$ and this is equal to $Tf = \bar{s}\tau(sf)$ if and only if $\bar{l}\tau(slf) = \tau(sf)$, or equivalently, if and only if $\tau(lsf) = l\tau(sf)$ for any $f \in L_1(\Gamma)$. Since any member g of $L_1(\Gamma)$ can be written as $g = sf$ for some $f \in L_1(\Gamma)$, we then have that $Tf = \bar{r}\tau(rf)$ for all $f \in L_1(\Gamma)$ if and only if $\tau(lg) = l\tau g$ for all $g \in L_1(\Gamma)$. This is equivalent to the fact that $\tau^*l = l$ a.e. on Γ, by Lemma 2.4.

4. The general ratio ergodic theorem.

The representation theorem given in the preceding section can be applied to simplify the proof of Chacon's general ergodic theorem. It is known that in the special case of a positive operator this theorem can be proved in a fairly transparent way [1]. It will be shown below that Theorem 3.1 enables one to reduce the general case to this special case.

We will start with a few preliminary lemmas and then recall the ratio ergodic theorem for a positive operator.

LEMMA 4.1. *Let $\{f_n, n \geqq 0\}$ be a sequence of (finite complex valued and measurable) functions. Assume that for each $\varepsilon > 0$ there exists a sequence $\{\varphi_n, n \geqq 0\}$ of functions such that $\lim_{n\to\infty} \varphi_n$ exists and is finite a.e. and such that $\int \{\limsup_{n\to\infty} |f_n - \varphi_n|\} \, d\mu < \varepsilon$. Then $\lim_{n\to\infty} f_n$ exists and is finite a.e.*

Proof. Let $\sigma = \lim_{N\to\infty} \{\sup_{n,m \geqq N} |f_n - f_m|\}$. Then $0 \leqq \sigma \leqq \infty$ a.e. and to prove the theorem it is enough to show that $\sigma = 0$ a.e., or equivalently that $\int \sigma \, d\mu = 0$. But this follows from the fact that $\sigma \leqq 2 \limsup_{n\to\infty} |f_n - \varphi_n|$.

LEMMA 4.2. *Let $A \in \mathscr{I}$ and $f \in L_1$. Then $\lim_{n\to\infty} \int_A \tau^n f \, d\mu$ exists.*

Proof. From the remarks preceding Lemma 2.2 it follows that $\int_A g\,d\mu=\int_A \tau g\,d\mu$ for all $g\in L_1(A)$. Also note that it is enough to prove this lemma for $f\in L_1^+$. But in this case the proof follows from $\int_A \tau^n f\,d\mu=\int_A \tau(\chi_A\tau^n f)\,d\mu\leqq\int_A \tau^{n+1}f\,d\mu\leqq\int_X f\,d\mu$.

LEMMA 4.3. *Let $A\in\mathscr{I}$ and $f\in L_1$. Then* $\lim_{m\to\infty}\lim_{n\to\infty}\int_A \tau^n(\chi_{A^c}\tau^m f)\,d\mu=0$.

Proof. Since $\int_A \tau^n(\chi_A\tau^m f)\,d\mu=\int_A \tau^m f\,d\mu$, we have that

$$\int_A \tau^n(\chi_{A^c}\tau^m f)\,d\mu = \int_A \tau^{n+m}f\,d\mu-\int_A \tau^m f\,d\mu.$$

Then the proof follows from the preceding lemma.

LEMMA 4.4. *Let $A\in\mathscr{I}$ and $f\in L_1$. Then* $\lim_{m\to\infty}\lim_{n\to\infty}\int_A \tau^n|\chi_{A^c}T^m f|\,d\mu=0$.

Proof. This is a corollary of Lemma 4.3, since $|\chi_{A^c}T^m f|\leqq\chi_{A^c}\tau^m|f|$.

DEFINITION 4.1. Let $S\colon L_1\to L_1$ be an operator. A sequence $\{p_n, n\geqq 0\}$ of non-negative (measurable) functions on X is called S-admissible if $f\in L_1$ and $|f|\leqq p_n$ for some $n\geqq 0$ imply that $|Sf|\leqq p_{n+1}$.

It is easy to see that a sequence is T-admissible if and only if it is τ-admissible; in what follows, such a sequence will be called an admissible sequence. Note that if $p\in L_1^+$ then $\{\tau^n p, n\geqq 0\}$ is an admissible sequence.

Also, if $f\in L_1$, $p\in L_1^+$ and $n\geqq 0$, we then let $r_n(f,p)=(\sum_{k=0}^n T^k f)/(\sum_{k=0}^n \tau^k p)$, and $\rho_n(f,p)=(\sum_{k=0}^n \tau^k f)/(\sum_{k=0}^n \tau^k p)$, defined a.e. on $\{x\mid \sum_{k=0}^n \tau^k p(x)>0\}$.

A central result in the ergodic theory of a positive contraction τ is the Chacon-Ornstein theorem, which states that $\lim_{n\to\infty}\rho_n(f,p)=\rho(f,p)$ exists and is finite a.e. on $\{x\mid \sum_{k=0}^\infty \tau^k p(x)>0\}$ [7], [2]. It is further known that $\chi_C\rho(f,p)$ is $\mathscr{I}$-measurable and $\int_A \rho(f,p)p\,d\mu=\lim_{n\to\infty}\int_A \tau^n f\,d\mu$ for all $A\in\mathscr{I}$ [3], [1]. More generally, if $f\in L_1$ and if $\{p_n, n\geqq 0\}$ is an admissible sequence then $\lim_{n\to\infty}\{(\sum_{k=0}^n \tau^k f)/(\sum_{k=0}^n p_k)\}$ exists and is finite a.e. on $\{x\mid \sum_{k=0}^\infty p_k(x)>0\}$. This is a special case of Chacon's general ratio ergodic theorem [4], [5], stated below as Theorem 4.1. The proof of this special case can be obtained [1] by a slight modification of a proof [2] of the Chacon-Ornstein theorem. Finally, recall a lemma that is a basic step in the proof of the results mentioned above. If $f\in L_1$ and $p\in L_1^+$ then $\lim_{n\to\infty}\{(T^n f)/(\sum_{k=0}^n \tau^k p)\}=0$ a.e. on $\{x\mid \sum_{k=0}^\infty \tau^k p(x)>0\}$ [7].

LEMMA 4.5. *Let $g\in L_1$, $p\in L_1^+$, $p>0$ a.e. and let $f=T^m g-g$ for some $m\geqq 0$. Then $\lim_{n\to\infty} r_n(f,p)$ exists and is finite a.e. and it vanishes a.e. on C.*

Proof. Since $r_n(f,p)=(T^{n+m}g-g)/(\sum_{k=0}^n \tau^k p)$ the existence and finiteness of $\lim_{n\to\infty} r_n(f,p)$ follows directly from the remarks just preceding this lemma. In fact this limit is equal to $(-g)/(\sum_{k=0}^\infty \tau^k p)$ a.e. Hence it vanishes a.e. on C, since $\sum_{k=0}^\infty \tau^k p=\infty$ a.e. on C.

THEOREM 4.1. *Let $T\colon L_1\to L_1$ be a contraction and let $\{p_n, n\geqq 0\}$ be an admissible sequence. Then for any $f\in L_1$, $\lim_{n\to\infty}\{(\sum_{k=0}^n T^k f)/(\sum_{k=0}^n p_k)\}$ exists and is finite a.e. on $\{x\mid \sum_{k=0}^\infty p_k(x)>0\}$.*

Proof. Let $p \in L_1$, $p > 0$ a.e. be a fixed function. From the remarks preceding Lemma 4.5 it follows that it is enough to show the existence and finiteness of $r(f, p) = \lim_{n\to\infty} r_n(f, p)$ a.e.

If $f \in L_1(\Gamma)$, with the notations of Theorem 3.1, then $Tf = \bar{s}\tau(sf)$ and hence $T^k f = \bar{s}\tau^k(sf)$ for all $k \geqq 0$, since $s\bar{s} = 1$ a.e. on $\Gamma \in \mathscr{I}$. Therefore $r_n(f, p) = \bar{s}\rho_n(sf, p)$ and the Chacon-Ornstein theorem gives the result.

If $f \in L_1(\Delta)$ and $\varepsilon > 0$ then, by Theorem 3.1, there exists $g \in L_1(\Delta)$ so that if $\phi = Tg - g$ then $\|f - \phi\| < \varepsilon$. Hence $r_n(f, p) = r_n(\phi, p) + r_n(f - \phi, p)$ and $r_n(\phi, p)$ converges to a finite limit a.e., by Lemma 4.5. Also, $|r_n(f - \phi, p)| \leqq \rho_n(|f - \phi|, p)$ a.e. Hence $\limsup_{n\to\infty} |r_n(f - \phi, p)| \leqq \rho(|f - \phi|, p)$ a.e. Since $\int \rho(|f - \phi|, p)p\, d\mu = \int_C \rho(|f - \phi|, p)p\, d\mu = \lim_{n\to\infty} \int_C \tau^n|f - \phi|\, d\mu < \varepsilon$, Lemma 4.1 (applied with the measure $p\, d\mu$) gives the result. In fact in this case $\lim_{n\to\infty} r_n(f, p) = 0$ a.e.

From the last two paragraphs it follows that the theorem is true for any $f \in L_1(C)$, since $C = \Gamma + \Delta$.

To complete the proof, let $f \in L_1$. Since $\sum_{k=0}^{\infty} |T^k f| \leqq \sum_{k=0}^{\infty} \tau^k |f| < \infty$ a.e. on D, it is clear that $\chi_D r_n(f, p)$ converges a.e. Hence it is enough to consider $\chi_C r_n(f, p)$. Let $\varepsilon > 0$ and choose $m \geqq 0$, by Lemma 4.4 so that $\lim_{n\to\infty} \int_C \tau^n |\chi_D T^m f|\, d\mu < \varepsilon$. Now letting $\phi = \chi_C T^m f$ we have that $r_n(\phi, p)$ converges a.e. and $\chi_C r_n(f, p) = \chi_C r_n(\phi, p) + \chi_C r_n(f - \phi, p)$. To apply Lemma 4.1 we again have to estimate the integral of $\limsup_{n\to\infty} |\chi_C r_n(f - \phi, p)|$. But $\chi_C r_n(f - \phi, p) = \chi_C r_n(f - T^m f, p) + \chi_C r_n(\chi_D T^m f, p)$ and $\lim_{n\to\infty} \chi_C r_n(f - T^m f, p) = 0$ a.e. by Lemma 4.5. Hence $\limsup_{n\to\infty} |\chi_C r_n(f - \phi, p)| = \limsup_{n\to\infty} |\chi_C r_n(\chi_D T^m f, p)| \leqq \chi_C \rho(|\chi_D T^m f|, p)$, as before. Then

$$\int \chi_C \rho(|\chi_D T^m f|, p)p\, d\mu = \lim_{n\to\infty} \int_C \tau^n |\chi_D T^m f|\, d\mu < \varepsilon$$

and Lemma 4.1 completes the proof.

The limit $r(f, p)$ whose existence has just been proved can be identified as follows.

THEOREM 4.2. *Let $s \in L_\infty(\Gamma)$ (hence $s = 0$ a.e. on $\Gamma^C = X - \Gamma$) be a function with the properties stated in Theorem 3.1. Let $f \in L_1$, $p \in L_1$, $p > 0$ a.e. Then for each $A \in \mathscr{I}$, $\nu(A) = \lim_{n\to\infty} \int_A sT^n f\, d\mu$ exists and defines a (complex) measure ν on the σ-field $\mathscr{I}$. This measure is absolutely continuous with respect to μ and its Radon-Nikodym derivative with respect to $p\, d\mu$ is $sr(f, p)$. More concisely, $sr(f, p)$ is $\mathscr{I}$-measurable and for each $A \in \mathscr{I}$, $\lim_{n\to\infty} \int_A sT^n f\, d\mu$ exists and is equal to*

$$\int_A sr(f, p)p\, d\mu.$$

Proof. First we show that $sr(f, p)$ is $\mathscr{I}$-measurable. Note that if $f \in L_1(C)$, then $r(f, p) = \bar{s}\rho(sf, p)$ and $sr(f, p) = \chi_\Gamma \rho(sf, p)$, which is $\mathscr{I}$-measurable. For a general $f \in L_1$, from the proof of Theorem 4.1 it follows that $\chi_C r(\chi_C T^n f, p)$ converges to $\chi_C r(f, p)$ as $n \to \infty$, in the norm topology of $L_1(X, \mathscr{F}, p\, d\mu)$. Hence $sr(f, p)$ is also $\mathscr{I}$-measurable.

Now let $A \in \mathscr{I}$. Then $\int_A sr(f, p)p\, d\mu = \lim_{n\to\infty} \int_A sr(\chi_C T^n f, p)p\, d\mu$. But

$$\int_A sr(\chi_C T^n f, p) = \int_A \chi_\Gamma \rho(s\chi_C T^n f, p)p\, d\mu = \int_{A\cap\Gamma} sT^n f\, d\mu = \int_A sT^n f\, d\mu$$

and the remaining statements of the theorem follow.

To note another property of $r(f, p)$ first we give two definitions.

Definition 4.2. Let $f, g \in L_1$ and $p \in L_1^+$, $p>0$ a.e. Then f and g are called equivalent ($f \sim g$) if $\lim_{n\to\infty} r_n(f-g, p)$ exists and is zero a.e. on C.

Note that this definition is independent of Theorem 4.1, for it is not assumed that the existence of $\lim_{n\to\infty} r_n(f-g, p)$ is known. Also note that the Chacon-Ornstein theorem implies that this notion of equivalence is independent of the choice of $p \in L_1^+$, $p>0$ a.e.

Definition 4.3. If $f \in L_1$ then $M(f) = \inf_{g\sim f} \|g\|$ is called the minimal norm of f. A function $f \in L_1$ is called minimal if $M(f) = \|f\|$.

Theorem 4.3. *Let $f \in L_1$, $p \in L_1^+$ and $p>0$ a.e. Then $\chi_C r(f, p)p \sim f$ and $\chi_C r(f, p)p$ is a minimal function.*

Proof. First we show that $r(f, p) = r(\chi_C r(f, p)p, p)$ a.e. on C, or equivalently, a.e. on Γ, since both of these functions are zero a.e. on $\Delta = C - \Gamma$. To this end it is enough to show that if $s \in L_\infty(\Gamma)$ is a function as in Theorem 4.2, then $sr(f, p) = sr(\chi_C r(f, p)p, p)$. Now, since both of these functions are $\mathscr{I}$-measurable, it is enough to check that they have the same integral on each $A \in \mathscr{I}$, with respect to the measure $p\, d\mu$. But $\chi_C r(f, p)p \in L_1(C)$, and this implies that $sr(\chi_C r(f, p)p, p) = \rho(s\chi_C r(f, p)p, p) = \rho(sr(f, p)p, p)$, as noticed in the proof of Theorem 4.1. Hence, for each $A \in \mathscr{I}$,

$$\begin{aligned}\int_A sr(\chi_C r(f, p)p, p)p\, d\mu &= \int_A \rho(sr(f, p)p, p)p\, d\mu \\ &= \lim_{n\to\infty} \int_A \tau^n(sr(f, p)p)\, d\mu = \int_A sr(f, p)p\, d\mu,\end{aligned}$$

where the second equality follows from the remarks preceding Lemma 4.5 and the last equality from the facts that $sr(f, p)p \in L_1(C)$ and that both A and $C-A$ are invariant sets. Hence $\chi_C r(f, p)p \sim f$.

To show that $\chi_C r(f, p)p$ is a minimal function we first show that $\|\chi_C r(f, p)p\| \leqq \|g\|$ or equivalently, $\|sr(f, p)p\| \leqq \|g\|$ for each $g \sim f$. But, since $sr(f, p)p = sr(g, p)p$, Theorem 4.2 implies that $\lim_{n\to\infty} \int_A sT^n g\, d\mu = \int_A sr(f, p)p\, d\mu$ for every $A \in \mathscr{I}$. It is then easy to see that $\lim_{n\to\infty} \int hsT^n g\, d\mu = \int hsr(f, p)p\, d\mu$ for each $\mathscr{I}$-measurable $h \in L_\infty(C)$, since the sequence $sT^n g$ is bounded in L_1-norm. Now find an $\mathscr{I}$-measurable $h \in L_\infty(C)$, so that $|h|=1$ a.e. on C and so that $hsr(f, p) = |sr(f, p)|$. This is possible, since $sr(f, p)$ is $\mathscr{I}$-measurable. Then, for this choice of h, $\lim_{n\to\infty} \int shT^n g\, d\mu = \|sr(f, p)p\|$ and hence $\|sr(f, p)p\| \leqq \liminf_{n\to\infty} \|shT^n g\| \leqq \|g\|$.

Now if $g \sim \chi_C r(f, p)p$ then it is easily seen that $g \sim f$ and hence $\|\chi_C r(f, p)p\| \leqq \|g\|$. Therefore $\chi_C r(f, p)p$ is a minimal function.

An immediate corollary of this theorem is

THEOREM 4.4. *For any $f \in L_1$ there exists a minimal function $\varphi \sim f$.*

5. **Minimal functions.** As noticed before the equivalence and minimality of functions are defined independently of Theorem 4.1 (and also, of Theorem 3.1). It may be instructive to observe that Theorem 4.4 can also be proved directly from the results of the ergodic theory of a positive contraction. This theorem then provides a starting point to obtain Theorem 4.1 and Theorem 3.1.

In this section we would like to sketch such a direct proof of Theorem 4.4, since it depends on a few results that seem intrinsically interesting. Before doing this, however, note that Theorem 4.4 implies Theorem 4.1 in a rather straightforward way. Let $f \in L_1$ be given and let $\varphi \sim f$ be a minimal function. Then, for each n, $m \geqq 0$, $T^n\varphi \sim \varphi$ and $\frac{1}{2}(T^n\varphi + T^m\varphi) \sim \varphi$, which imply that

$$\|\tfrac{1}{2}(T^n\varphi+T^m\varphi)\| = \tfrac{1}{2}(\|T^n\varphi\|+\|T^m\varphi\|) = \|\varphi\|.$$

Hence, if $E = \bigcup_{n=0}^{\infty} \{x \mid T^n\varphi(x) \neq 0\}$, then there exists a function $s \in L_\infty(E)$ such that $|s| = 1$ a.e. on E and such that $T^n\varphi = \bar{s}|T^n\varphi|$ for all $n \geqq 0$. Then one can see that $T^n\varphi = \bar{s}\tau^n|\varphi| = \bar{s}\tau^n(s\varphi)$ and the existence of $r(\varphi, p) = \bar{s}\rho(s\varphi, p)$ follows. Hence $r(f, p) = r(\varphi, p) + r(f-\varphi, p)$ also exists.

We now start the proof of Theorem 4.4.

DEFINITION 5.1. For a fixed $f \in L_1$ and for a given sequence $\{h_n, n \geqq 0\}$ of L_∞-functions, satisfying $0 \leqq h_n \leqq 1$ for all $n \geqq 0$, we let

$$\begin{aligned} a_0 &= h_0 f, & b_0 &= (1-h_0)f, \\ a_n &= h_n T b_{n-1}, & b_n &= (1-h_n)Tb_{n-1}, \quad n \geqq 1, \end{aligned}$$

and

$$\begin{aligned} \alpha_0 &= h_0|f|, & \beta_0 &= (1-h_0)|f| \\ \alpha_n &= h_n \tau \beta_{n-1}, & \beta_n &= (1-h_n)\tau\beta_{n-1}, \quad n \geqq 1. \end{aligned}$$

Note that $|a_n| \leqq \alpha_n$, $|b_n| \leqq \beta_n$ and also that $\sum_{k=0}^{n} \|\alpha_k\| + \|\beta_n\| \leqq \|f\|$ for all $n \geqq 0$, as simple induction arguments show. Hence $\sum_{k=0}^{\infty} a_k = a$ and $\sum_{k=0}^{\infty} \alpha_k = \alpha$ exist a.e. and also in the L_1-norm. Also, $\|a\| \leqq \|\alpha\| \leqq \|f\|$. It is clear that $\sum_{k=0}^{n} a_k + b_n \sim f$, since $Tf \sim f$.

LEMMA 5.1. *If $\lim_{m\to\infty} \lim_{n\to\infty} \int_C \tau^n \beta_m \, d\mu = 0$, then $a \sim f$.*

Proof. Let $l = \limsup_{n\to\infty} |r_n(f-a, p)|$. Then for all $m \geqq 0$,

$$\begin{aligned} l &= \limsup \left| r_n\left(\sum_{k=0}^{m} a_k + b_m - a, p \right) \right| \\ &\leqq \rho\left(\left| \sum_{k=0}^{m} a_k - a \right|, p \right) + \rho(|b_m|, p) \leqq \rho\left(\left| \sum_{k=0}^{m} a_k - a \right|, p \right) + \rho(\beta_m, p). \end{aligned}$$

Hence $\int_C lp \, d\mu \leqq \|\sum_{k=0}^{m} a_k - a\| + \lim_{n\to\infty} \int_C \tau^n \beta_m \, d\mu$, which can be made arbitrarily small if $m \geqq 0$ is sufficiently large. Therefore $l = 0$ a.e. on C.

LEMMA 5.2. *If $h_n=\chi_C$ for all $n\geqq 0$, then $a\sim f$.*

Proof. In this case we can see that $\beta_n=\chi_D\tau^n|f|$, since $\chi_D\tau g=0$ for any $g\in L_1(C)$. Then the proof follows from Lemma 4.3.

LEMMA 5.3. *Given $f\in L_1$ there exists $f'\in L_1(C)$ such that $f'\sim f$ and $\|f'\|\leqq\|f\|$.*

Proof. This is a corollary of Lemma 5.2.

LEMMA 5.4. *Let $f_1, f_2\in L_1(C)$ and $f_1\sim f_2\sim f$. Let $A_1\in\mathscr{I}$, $A_2=C-A_1$. Then $\chi_{A_1}f_1+\chi_{A_2}f_2\sim f$.*

Proof. This is a consequence of the fact that if $g\in L_1(C)$ and $A\in\mathscr{I}$ then $T^n\chi_A g=\chi_A T^n g$.

DEFINITION 5.2. For a real valued L_1-function λ we let $U\lambda=\tau\lambda^+-\lambda^-$ [9].

Note that if $\lambda\leqq l$ then $U\lambda\leqq Ul$. In particular $(U^n\lambda)^+\leqq(U^n l)^+$ for all $n\geqq 0$.

LEMMA 5.5. *Given $f, g\in L_1^+$ there exists a sequence $\{h_n, n\geqq 0\}$ of L_∞-functions such that $0\leqq h_n\leqq 1$ for all $n\geqq 0$ and such that*

$$[U^n(f-g)]^+ = \beta_n, \qquad [U^n(f-g)]^- = g-\sum_{k=0}^{n}\alpha_k,$$

with the notations of Definition 5.1.

Proof. Follows by an induction on $n\geqq 0$.

Hence, if $\{h_n, n\geqq 0\}$ is defined as in Lemma 5.5, then $0\leqq|a|\leqq\alpha\leqq g$.

We now recall a maximal ergodic theorem for a positive contraction, involving α and g.

THEOREM 5.1. *If $\sup_{n\geqq 0}\sum_{k=0}^{n}\tau^k(f-g)>0$ a.e. on $E\in\mathscr{F}$ then $\alpha=g$ a.e. on E, with the same notations as above.*

The proof can be found in [7], [2], [1].

LEMMA 5.6. *Let $f, g\in L_1^+(C)$ and let $\int_A f\,d\mu\leqq\int_A g\,d\mu$ for each $A\in\mathscr{I}$. Then $\lim_{n\to\infty}\|\{U^n(f-g)\}^+\|=0$.*

Proof. We know that $\int_A f\,d\mu=\int_A \rho(f,g)g\,d\mu$ for each $A\in\mathscr{I}$. Since $\rho(f,g)$ is $\mathscr{I}$-measurable, this means that $0\leqq\rho(f,g)\leqq 1$ a.e. Also, $\rho(\rho(f,g)g, g)=\rho(f,g)$. Hence, if $0\leqq\lambda<1$ is any number then $\sup_{n\geqq 0}\sum_{k=0}^{n}\tau^k(f-\lambda\rho(f,g)g)>0$ a.e. on $\{x \mid \rho(f,g)(x)>0\}$.

Now let α_n, β_n, α be defined as in Lemma 5.5 for the functions f and $\lambda\rho(f,g)g$, so that $[U^n(f-\lambda\rho(f,g)g)]^+=\beta_n$ and $[U^n(f-\lambda\rho(f,g)g)]^-=\lambda\rho(f,g)g-\sum_{k=0}^{n}\alpha_k$. Then $0\leqq\alpha\leqq\lambda\rho(f,g)g$ and also $\alpha=\lambda\rho(f,g)g$ a.e. on $\{x \mid \rho(f,g)(x)>0\}$, by Theorem 5.1. Hence $\alpha=\lambda\rho(f,g)g$. Therefore,

$$\lim_{n\to\infty}\|\beta_n\| \leqq \|f\|-\|\alpha\| = \|f\|-\|\lambda\rho(f,g)g\| = (1-\lambda)\|f\|.$$

Since $\|\{U^n(f-g)^+\|\leqq\|\beta_n\|$ for each λ, $0\leqq\lambda<1$, the proof follows.

THEOREM 5.2. *Let $f \in L_1(C)$ and $g \in L_1^+$ so that $\int_A |f|\, d\mu \leqq \int_A g\, d\mu$ for each $A \in \mathscr{I}$. Then there exists $a \sim f$ so that $|a| \leqq g$ and that $\|a\| \leqq \|f\|$.*

Proof. Let α_n, β_n be defined as in Lemma 5.5 for the functions $|f|$ and g so that

$$[U^n(|f|-g)]^+ = \beta_n, \qquad [U^n(|f|-g)]^- = g - \sum_{k=0}^{n} \alpha_k.$$

Let a_n, b_n, a be the associated functions as in Definition 5.1. Then Lemma 5.6 shows that $\lim_{n\to\infty} \|\beta_n\| = 0$ and Lemma 5.1 gives that $a \sim f$. Then $|a| \leqq \alpha \leqq g$ is clear.

Proof of Theorem 4.4. Let $f \in L_1$ and let $f_n \sim f$ be a sequence in L_1 so that $\lim_{n\to\infty} \|f_n\| = M(f)$, the minimal norm of f. By Lemma 5.3 we may and will assume that $f_n \in L_1(C)$ for each $n \geqq 0$.

We now define a new sequence g_n as follows. Let $g_0 = f_0$. Let

$$A_1 = \{x \mid \rho(|f_1|, |g_0|)(x) < 1\} \cap C, \qquad B_1 = C - A_1.$$

Let $g_1' = \chi_{A_1} f_1 + \chi_{B_1} g_0$. Then by Lemma 5.4, $g' \sim f$ and also, by the construction, that $\int_A |g_1'|\, d\mu \leqq \int_A |g_0|\, d\mu$. Hence, by Theorem 5.2 there is $g_1 \sim f$ so that $|g_1| \leqq |g_0|$. It is also clear that $\|g_1\| \leqq \|f_1\|$.

Continuing this process, we define

$$A_n = \{x \mid \rho(|f_n|, |g_{n-1}|)(x) < 1\} \cap C, \quad B_n = C - A_n, \quad g_n' = \chi_{A_n} f_n + \chi_{B_n} g_{n-1}.$$

Then $g_n' \sim f$ and $\|g_n'\| \leqq \|f_n\|$ and also, $\int_A |g_n'|\, d\mu \leqq \int_A |g_{n-1}|\, d\mu$ for each $A \in \mathscr{I}$. Hence there exists $g_n \sim f$ so that $|g_n| \leqq |g_{n-1}|$.

Therefore we now have a sequence $g_n \in L_1(C)$, $n \geqq 0$, so that $g_n \sim f$, $\lim_{n\to\infty} \|g_n\| = M(f)$ and $|g_n| \leqq |g_{n-1}|$, $n \geqq 1$.

It will be shown that g_n is a Cauchy sequence. Let $0 \leqq n \leqq m$ and let $2\theta_{mn}(x)$ be the angle between the complex numbers $g_n(x)$ and $g_m(x)$, with $0 \leqq \theta_{nm}(x) \leqq \pi/2$. Then $|g_m - g_n| \leqq 2|g_n| \sin\theta_{nm} + [|g_n| - |g_m|]$ and $|\frac{1}{2}(g_n+g_m)| \leqq |g_n| \cos\theta_{nm}$, as one can see easily. But $\frac{1}{2}(g_n+g_m) \sim f$, and this means that $M(f) \leqq \int |g_n| \cos\theta_{nm}\, d\mu$. Since $\lim_{n\to\infty} \int |g_n|\, d\mu = M(f)$ we then have that $\int |g_n| \sin\theta_{nm}\, d\mu \to 0$ as $n, m \to \infty$, $n \leqq m$. Hence we see that $\|g_n - g_m\| \to 0$ as $n, m \to \infty$.

Let $\lim_{n\to\infty} g_n = \varphi$, in the L_1-norm. Then it is clear that $\|\varphi\| = M(f)$. To complete the proof it is now enough to show that $\varphi \sim f$. But, if $l = \limsup_{n\to\infty} |r_n(f-\varphi, p)| = \limsup_{n\to\infty} |r_n(g_m - \varphi, p)|$, then $l \leqq \rho(|g_m - \varphi|, p)$ for each $m \geqq 0$, and hence $\int lp\, d\mu \leqq \|g_m - \varphi\|$, which shows that $l = 0$ a.e. on C.

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